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A Refinement of an Lp Inequality for a Polynomial with Restrictions on Zeros

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Abstract

If Pn denotes a class of polynomials of degree n, then for $P/inPn$, the Lp analogue of Bernstein's inequality was proved by Zygmund. In fact, he proved that

$$\|P'\|_p \leq n\|P\|_p, \quad for \;\; p > 0.$$

In literature, so many generalizations and refinements of this result exists. Recently K Krishnadas and B Chanam [10,Theorem 1] proved a generalisation of above result. In this paper we prove a refinement of above result which in turn provides a generalization of several other results.

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1 Introduction and statement of results

Let P_n denotes the class of polynomials $P(z) = \sum_{j=0}^{n} a_v z^v$ of degree n.

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For $P \in P_n$ define,

$$\|P\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta)}|^p d\theta \right\}^{\frac{1}{p}}, \quad p > 0$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|$$

Further

$$\|P\|_0 := exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} log|P(e^{i\theta}|d\theta \right\}$$

If $P \in P_n$, Zygmund [18] proved that

$$\|P'\|_p \leq n\|P\|_p, \quad p > 0 \tag{1.1}$$

Letting $p \to \infty$, we get

$$\|P'\|_\infty \leq n\|P\|_\infty \tag{1.2}$$

Inequality (1.2) is known as Bernstein's inequality where equality occurs for $P(z) = \alpha z^n, \alpha \neq 0$.
The validity of (1.1) for $0 < p < 1$ was proved by Arestov [1].
If $P \in P_n$ such that $P(z) \neq 0$ in $|z| < 1$, then inequalities (1.2) and (1.1) can be replaced respectively by

$$\|P'\|_\infty \leq \frac{n}{2}\|P\|_\infty \tag{1.3}$$

and

$$\|P'\|_p \leq \frac{n}{\|1 + z\|_p}\|P\|_p, \quad p > 0. \tag{1.4}$$

While inequality (1.3) was conjectured by Erdös and later proved by Lax [10], inequality (1.4) was proved by Debruijn [4] for $p \geq 1$. Rahman and Schmeisser[14] showed that (1.4) remain true for $0 < p < 1$.
On the other hand, if $p \in P_n$ has all its zeros in $|z| \leq 1$, Turan[18] proved that

$$\|P'\|_\infty \geq \frac{n}{2}\|P\|_\infty. \tag{1.5}$$

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Both inequalities (1.3) and (1.5) attain equality for the polynomial $P(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.
Govil and Rahman [7] extended (1.4) into Lp version by proving that, if $P \in Pn$ has no zero in $|z| < k, k \geq 1$, then

$$\|P'\|_p \leq \frac{n}{\|k+z\|_p}\|P\|_p, \quad p \geq 1 \tag{1.6}$$

Gardener and Weems [6] and Rather [14] independently proved that (1.6) also holds for $0 < p < 1$.
Also, Aziz and Rather[2] proved that if $P \in P_n$ has no zero in $|z| < k, k \geq 1$, then for each $p > 0$

$$\|P'\|_p \leq \frac{n}{\|\delta_{k,1}+z\|_p}\|P\|_p, \quad p \geq 1 \tag{1.7}$$

where $\delta_{k,1} = \dfrac{n|a_0|k^2 + k^2|a_1|}{n|a_0| + k^2|a_1|}$

Malik[12] proved in the sense that the inequality involves integral mean of $|P(z)|$ on $|z| = 1$ and extends to Lp norm. He proved that if $P \in P_n$ has all its zeros in $|z| \leq 1$, then for each $p > 0$.

$$\|P'\|_p \geq \frac{n}{\|1+z\|_p}\|P\|_p. \tag{1.8}$$

As a generalisation of (1.8), Aziz and Rather [2] proved that if $P \in P_n$ has all its zeros in $|z| \leq k, k \leq 1$, then for each $p > 0$

$$\|p'\|_p \geq \frac{n}{\|1+t_{k,1}z\|_p}\|p\|_p. \tag{1.9}$$

and

$$\|p'\|_\infty \geq \frac{n}{\|1+t_{k,1}z\|_p}\|p\|_p. \tag{1.10}$$

where $t_{k,1} = \dfrac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|}$

Several improvements, generalizations and extensions of the estimates of the bounds of $\left\{\dfrac{\|P'\|_\infty}{\|P\|_\infty}\right\}$ under prescribed restrictions on zeros of $P(z)$ are

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available in literature. It is also interesting to check the dependence of the $\left\{\dfrac{\|P'\|_\infty}{\|P\|_\infty}\right\}$ on the coefficients of the polynomial under consideration. In this direction Govil et al. [8] proved the following two results, where they improved (1.6) and (1.8), and the second (1.11) by involving certain coefficients of the polynomial.

Theorem 1.1. If $P \in P_n$ has no zeros in $|z| < k, k \geq 1$, then

$$\|P'\|_\infty \leq \frac{n}{1+k}\frac{(1-|\lambda|)(1+k^2|\lambda|)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(1-k+k^2+k|\lambda|)+k(n-1)|\mu-\lambda^2|}\|P\|_\infty, \tag{1.11}$$

where

$$\lambda = \frac{ka_1}{na_0} \text{ and } \mu = \frac{2k^2a_2}{n(n-1)a_0}.$$

Theorem 1.2. If $P \in P_n$ has all its zeros in $|z| \leq k, k \leq 1$, then

$$\|P'\|_\infty \geq \frac{n}{1+k}\frac{(1-|\omega|)(1+k^2|\omega|)+k(n-1)|\gamma-\omega^2|}{(1-|\omega|)(1-k+k^2+k|\omega|)+k(n-1)|\gamma-\omega^2|}\|P\|_\infty, \tag{1.12}$$

where

$$\omega = \frac{\overline{a}_{n-1}}{nk\overline{a}_n} \text{ and } \gamma = \frac{2\overline{a}_{n-2}}{n(n-1)k^2\overline{a}_n}.$$

Recently, Krishnadas and Chanam[10] proved the following two results where they extended the inequalities (1.14) and (1.15) to Lp norms. ,

Theorem 1.3. If $P \in Pn$ has no zeros in $|z| < k, k \geq 1$, then for each $p > 0$

$$\|P'\|_p \leq \frac{n}{\|C+z\|_p}\|P\|_p. \tag{1.13}$$

where

$$C = k\frac{(1-|\lambda|)(|\lambda|+k^2)+k(n-1)|\mu-\lambda^2|}{(1-|\lambda|)(|\lambda|k^2+1)+k(n-1)|\mu-\lambda^2|} \tag{1.14}$$

$$\lambda = \frac{ka_1}{na_0} \text{ and } \mu = \frac{2k^2a_2}{n(n-1)a_0}.$$

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Theorem 1.4. If $P \in Pn$ has all its zeros in $|z| \leq k, k \leq 1$, then for each $p > 0$

$$\|P'\|_p \geq \frac{n}{\|1 + \mathcal{D}z\|_p} \|P\|_p. \tag{1.15}$$

where

$$\mathcal{D} = k \frac{(1 - |\omega|)(|\omega| + k^2) + k(n-1)|\gamma - \omega^2|}{(1 - |\omega|)(|\omega|k^2 + 1) + k(n-1)|\gamma - \omega^2|} \tag{1.16}$$

$$\omega = \frac{\overline{a}_{n-1}}{nk\overline{a}_n} \text{ and } \gamma = \frac{2\overline{a}_{n-2}}{n(n-1)k^2\overline{a}_n}.$$

2 Main Results

In this paper,first we obtain the following result which includes not only a refinement of Theorem 1.3 but also provides some generalization of other results.

Theorem 2.1. If $P \in Pn$ has no zeros in $|z| < k, k \geq 1$, then for each $p > 0$

$$\|P' + \beta nmz^{n-1}\|_p \leq \frac{n}{\|C + z\|_p} \|P\|_p. \tag{2.1}$$

where C is defined by (1.17) and $m = \underset{|z|=k}{min} |P(z)|$.

Remark 2.2 If we put $m = 0$ and let $p \to \infty$ in (2.1), the Theorem 2.1 reduces to Theorem 1.1 by using the inequality (1.17).

Remark 2.3 For $m = 0$, inequality (2.1) reduces to inequality (1.16). Next, we prove the theorem as a refinement of Theorem 1.4. In fact we prove

Theorem 2.4. If $P \in Pn$ has all its zeros in $|z| \leq k, k \leq 1$, then for each $p > 0$

$$\|P' + \beta nmz^{n-1}\|_p \geq \frac{n}{\|1 + \mathcal{D}z\|_p} \|P\|_p. \tag{2.2}$$

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where $\mathcal{D}$ is defined by (1.19)

Remark 2.5. For $m = 0$ Theorem 2.4 reduces to Theorem 1.4.

Instead of proving Theorem 2.4, we prove a more general result, from which Theorem 2.4,follows as a special case.

Theorem 2.6. If $P \in P_n$ has all its zeros in $|z| \leq k, k \leq 1$, then for each $\delta > 0, r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$.

$$\|P' + \beta nmz^{n-1}\|_{sp} \geq \frac{n}{\|1 + \mathcal{D}z\|_{rp}}\|P\|_p. \tag{2.3}$$

where $\mathcal{D}$ is defined by (1.19).

Remark 2.7. Put $r = 0$ or $s = 0$, we obtain Theorem 2.4.

3 Lemmas

In this section we present some lemmas which will help us to prove our results.

Lemma 3.1. If $P \in P_n$ has no zeros in $|z| < k, k \geq 1$, then

$$C|P'(z)| \leq |Q'(z)| \tag{3.1}$$

where C is defined by (1.17) and $Q(z) = z^n\overline{P(\frac{1}{\overline{z}})}$

Above Lemma 3.1 is due to Govil et al.[8]. By applying Lemma (3.1.) to the polynomial $F(z) = P(z) + m\beta z^n$, we immediately get the following result.

Lemma 3.2. If $P \in P_n$ has no zeros in $|z| < k, k \geq 1$, then for any complex number β with $|\beta| \leq 1$,

$$C|P'(z) + \beta nmz^{n-1}| \leq |Q'(z)| \tag{3.2}$$

where C is defined by (1.17) and $m = \min_{|z|=k}|P(z)|$.

Lemma 3.3. If $P \in P_n$ has all its zeros in $|z| \leq k, k \leq 1$, then on $|z| = 1$

$$|Q'(z)| \leq \mathcal{D}|P'(z) + nm\beta z^{n-1}| \tag{3.3}$$

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where $\mathcal{D}$ is defined by (1.19).

Proof of Lemma 3.3. Since P(z) has all its zeros in $|z| \leq k, k \leq 1$, then the polynomial $Q(z) = z^n\overline{P}(\frac{1}{\overline{z}})$ has no zeros in $|z| < \frac{1}{k}, \frac{1}{k} \geq 1$. Thus aapplying Lemma 3.2 to the polynomial Q(z), we have

$$|Q'(z)| \leq k\frac{(1-|\omega|)(|\frac{1}{k^2}\omega|+1)+\frac{1}{k}(n-1)|\gamma-\omega^2|}{(1-|\omega|)(|\omega|+k^2)+\frac{1}{k}(n-1)|\gamma-\omega^2|}\left|P'(z)+nm\beta z^{n-1}\right|$$

$\omega = \frac{1/k}{n}\frac{\overline{a}_{n-1}}{\overline{a}_n} = \frac{\overline{a}_{n-1}}{nk\overline{a}_n}$ and $\gamma = \frac{2/k^2}{n(n-1)k^2}\frac{\overline{a}_{n-2}}{\overline{a}_n} = \frac{2\overline{a}_{n-2}}{n(n-1)k^2\overline{a}_n}$.
Then,

$$|Q'(z)| \leq k\frac{(1-|\omega|)(|\omega|+k^2)+k(n-1)|\gamma-\omega^2|}{(1-|\omega|)(|\omega|k^2+1)+k(n-1)|\gamma-\omega^2|}\left|P'(z)+nm\beta z^{n-1}\right|$$

which proves Lemma 3.3.

Lemma 3.4. If $P \in P_n$, then for each α, $0 \leq \alpha < 2\pi$ and $p > 0$

$$\int_0^{2\pi}\int_0^{2\pi}|Q'(e^{i\theta})+e^{i\alpha}P'(e^{i\theta})|^p d\theta d\alpha \leq 2\pi n^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta \tag{3.4}$$

The above lemma is due to Aziz [2]

Lemma 3.5. If $P \in P_n$, then for each α, $0 \leq \alpha < 2\pi$,$p > 0$ and for any complex number β with $|\beta| \leq 1$,

$$\int_0^{2\pi}\int_0^{2\pi}|Q'(e^{i\theta})+e^{i\alpha}\left\{P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right\}|^p d\theta d\alpha \leq 2\pi n^p\int_0^{2\pi}|P(e^{i\theta})|^p d\theta \tag{3.5}$$

where $m = \min_{|z|=k}|P(z)|$.
Proof of Lemma 3.5. By applying Lemma (3.4.) to the polynomial $F(z) = P(z) + m\beta z^n$, we can easily get the proof of Lemma 3.5.

Lemma 3.6. Let z be any complex and independent of α , where α is any real, then for $p > 0$

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$$\int_0^{2\pi} |1+ze^{i\alpha}|^p d\alpha = \int_0^{2\pi} |e^{i\alpha}+|z||^p d\alpha \tag{3.6}$$

Lemma 3.6 is due to Gardner and Govil[5].

4 Proof of Theorems

Proof of Theorem 2.1. Since P(z) has no zeros in $|z|<k, k\geq 1$, we have by Lemma 3.2.

$$C|P'(z)+\beta nmz^{n-1}| \leq |Q'(z)| \tag{4.1}$$

where C is defined by (1.17) and $m=\underset{|z|=k}{min}|P(z)|$.
We have for every real α and $\mathcal{R}\geq r\geq 1$,

$$|\mathcal{R}+e^{i\alpha}| \geq |r+e^{i\alpha}|$$

Then, for every $p>0$, we have

$$\int_0^{2\pi} |\mathcal{R}+e^{i\alpha}|^p d\alpha \geq \int_0^{2\pi} |r+e^{i\alpha}|^p d\alpha \tag{4.2}$$

For points $e^{i\theta}$, $0\leq\theta<2\pi$, for which $P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\neq 0$, we denote $\mathcal{R}=\left|\dfrac{Q'(e^{i\theta})}{P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}}\right|$ and $r=C$, then by (4.1), we for each $p>0$

$$\begin{aligned}
&\int_0^{2\pi} \left|Q'(e^{i\theta})+e^{i\alpha}\left\{P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right\}\right|^p d\alpha \\
&= \left|P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right|^p \int_0^{2\pi} \left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}}+e^{i\alpha}\right|^p \\
&= \left|P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right|^p \int_0^{2\pi} \left|\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}}\right|+e^{i\alpha}\right|^p \qquad by \quad (3.6) \\
&= \left|P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right|^p \int_0^{2\pi} \left|\mathcal{R}+e^{i\alpha}\right|^p \\
&\geq \left|P'(e^{i\theta})+nm\beta e^{i(n-1)\theta}\right|^p \int_0^{2\pi} \left|r+e^{i\alpha}\right|^p \qquad by \quad (4.2)
\end{aligned}$$

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Hence,

$$\int_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\alpha}\left\{P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}\right\}\right|^p d\alpha \geq \left|P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}\right|^p \int_0^{2\pi} \left|C + e^{i\alpha}\right|^p \tag{4.3}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta} \neq 0$.
For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta} = 0$, inequality (4.3) trivially holds. Hence, using (4.3) in Lemma 3.5, we obtain for each $p > 0$,

$$\int_0^{2\pi} \left|P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}\right|^p d\theta \int_0^{2\pi} \left|C + e^{i\alpha}\right|^p \leq 2\pi n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta$$

which is equivalent to

$$\left\{\int_0^{2\pi} \left|P'(e^{i\theta}) + nm\beta e^{i(n-1)\theta}\right|^p d\theta\right\}^{\frac{1}{p}} \leq n\left\{\frac{1}{2\pi}\int_0^{2\pi} \left|C + e^{i\alpha}\right|^p d\alpha\right\}^{-\frac{1}{p}}\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \tag{4.4}$$

from which the desired conclusion of Theorem 2.1 follows.

Proof of Theorem 2.6. Since P(z) has all its zeros in $|z| \leq k, k \leq 1$, $P(z)$ also has all its zeros in $|z| \leq k, k \leq 1$. Hence, by Gauss-Lucas Theorem, the polynomial

$$z^{n-1}\overline{P(\frac{1}{\overline{z}})} = nQ(z) - zQ'(z) \tag{4.5}$$

has all its zeros in $|z| \geq \frac{1}{k}, \frac{1}{k} \geq 1$. Further, since P(z) has all its zeros in $|z| \leq k, k \leq 1$, we have by Lemma 3.3.

$$\begin{aligned} |Q'(z)| &\leq \mathcal{D}|P'(z) + nm\beta z^{n-1}| \\ &= \mathcal{D}\left\{|P'(z)| + tnm\right\} \quad for \quad |z| = 1. \end{aligned} \tag{4.6}$$

where $\mathcal{D}$ is defined by (1.19) and $|\beta| = t$.
For $|z| = 1$, we also have

$$|P'(z) = |nQ(z) - zQ'(z)|. \tag{4.7}$$

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Using (4.7) in (4.6) , we have on $|z| = 1$

$$|Q'(z)| \leq \mathcal{D}\left\{|nQ(z) - zQ'(z)| + tnm\right\} \tag{4.8}$$

Thus, in view of (4.5) and (4.8), the function

$$\phi(z) = \frac{zQ'(z)}{\mathcal{D}\left\{|nQ(z) - zQ'(z)| + tnm\right\}}$$

is analytic in $|z| \leq 1, |\phi(z)| \leq 1$ on $|z| = 1$ and $\phi(0) = 0$. Therefore, the function $1 + \mathcal{D}\phi(z)$ is subordinate to the function $1 + \mathcal{D}z$ for $|z| \leq 1$. Hence, by a well known property of subordination [9], we have for each $p > 0$

$$\int_0^{2\pi} |1 + \mathcal{D}\phi(e^{i\theta})|^p d\theta \geq \int_0^{2\pi} |1 + \mathcal{D}e^{i\theta}|^p d\theta \tag{4.9}$$

Now,

$$1 + \mathcal{D}\phi(z) = 1 + \frac{zQ'(z)}{|nQ(z) - zQ'(z)| + tnm} = \frac{nQ(z)}{|nQ(z) - zQ'(z)| + tnm}$$

which implies for $|z| = 1$,

$$\begin{aligned}|nQ(z)| =& |1 + \mathcal{D}\phi(z)| \, |nQ(z) - zQ'(z)| + tnm \\ &= |1 + \mathcal{D}\phi(z)| \, |P'(z)| + tnm \quad by \quad (4.7)\end{aligned}$$

Since $|P(z)| = |Q(z)|$ on $|z| = 1$, we have from the preceding inequality

$$n|P(z)| = |1 + \mathcal{D}\phi(z)| \, |P'(z)| + tnm \quad on \quad |z| = 1 \tag{4.10}$$

Then for each $p > 0$ and $0 \leq \theta < 2\pi$, we have

$$n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta = \int_0^{2\pi} |1 + \mathcal{D}\phi(e^{i\theta})|^p \left\{\left|P'(e^{i\theta})\right| + tnm\right\}^p d\theta$$

Applying Holder's inequality to the above inequality, we have for
$r > 1, s > 1$ with $r^{-1} + s^{-1} = 1$ and for each $p > 0$

$$n^p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta \leq \left\{\int_0^{2\pi} \left|1 + \mathcal{D}\phi(e^{i\theta})\right|^{rp} d\theta\right\}^{\frac{1}{r}} \left\{\int_0^{2\pi} \left\{\left|P'(e^{i\theta})\right| + tnm\right\}^{sp} d\theta\right\}^{\frac{1}{s}}$$

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which implies

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\dfrac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+\mathcal{D}\phi(e^{i\theta})\right|^{rp} d\theta\right\}^{\dfrac{1}{pr}}\left\{\int_0^{2\pi}\left\{\left|P'(e^{i\theta})\right|+tnm\right\}^{sp} d\theta\right\}^{\dfrac{1}{sp}}$$

using (4.9) in the above inequality , we have

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\dfrac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+\mathcal{D}e^{i\theta}\right|^{rp} d\theta\right\}^{\dfrac{1}{pr}}\left\{\int_0^{2\pi}\left\{\left|P'(e^{i\theta})\right|+tnm\right\}^{sp} d\theta\right\}^{\dfrac{1}{sp}}$$

by choosing argument of β as in the proof of Theorem 2.4 , we get the above inequality as

$$n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta\right\}^{\dfrac{1}{p}} \leq \left\{\int_0^{2\pi}\left|1+\mathcal{D}e^{i\theta}\right|^{rp} d\theta\right\}^{\dfrac{1}{pr}}\left\{\int_0^{2\pi}\left|P'(e^{i\theta})+\beta nme^{i(n-1)\theta}\right|^{sp} d\theta\right\}^{\dfrac{1}{sp}}$$

which is the desired conclusion of the theorem.

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