Some coefficient inequities for $\mu-$uniformly convex functions

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Abstract— In this present paper we define the class $C_p(\xi,\mu)$ and introduce and investigate coefficient estimates. In addition we provide conditions such that the confluent hypergeometric function, belongs to $C_p(\xi,\mu)$.

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INTRODUCTIONS

We show the set of all regular functions f in the unit disk $\Theta=\{z:|z|<1\}$ which are

$$f(z)=z+\sum_{\tau=2}^{\infty}a_\tau z^\tau \tag{1.1}$$

with A and let S be the subclass of A consisting of regular functions. Suppose that T be the subclass of S which are in the form

$$\varphi(z)=z-\sum_{\tau=2}^{\infty}a_\tau z^\tau \tag{1.2}$$

satisfies the conditions $a_\tau \geq 0$ $(\tau=2,3,\ldots)$ with $\sum_{\tau=2}^{\infty}a_\tau<1.$

Also suppose that $C^*(\xi)$ be the famous subclass of S which are convex of order ξ. Indeed $h\in C^*(\alpha)$ is equivalent to $\Re\left(1+\frac{zh''(z)}{h'(z)}\right)>\xi$ in Θ. This subclass has so long history in geometric function theory (for example see [2,3,6]).

Let $l,m,n\in C$ (the set of all complex numbers), such that $n\not\in 0,-1,-2,\ldots$. It is well known that the answer of the ordinary equation

$$(1-z)z\varphi''(z)+[n-z(l+m+1)]\varphi'(z)-lm\varphi(z)=0$$

is

$$\mathrm{F}\,(l,m,n;z)=\sum_{\tau=0}^{\infty}\frac{(l)_\tau(m)_\tau}{(n)_\tau(1)_\tau}z^\tau$$

and the function $\varphi(z)=z\mathrm{F}\,(l,m,n;z),\, z\in\Theta,$ is called hypergeometric function. We note that $(l)_0=1$ for $l\neq 0$ and $(l)_\tau=l(l+1)(l+2)\mathrm{K}\,(l+\tau-1).$

The hypergeometric function plays an important role in various fields. We refer to [5,7] and references therein for more details about this function.

Finally, for $-1\le\xi\le 1$ and $\tau\ge 0,$ we introduce a subclass $\mathrm{C}_p(\xi,\mu)$ of convex functions in the following way

$$\mathrm{C}_p(\xi,\mu)=\left\{\varphi\in\mathrm{S}:\Re\left(1+\frac{z\varphi''(z)}{\varphi'(z)}\right)\ge\mu\left|\frac{z\varphi''(z)}{\varphi'(z)}-1\right|+\xi,\, z\in\Theta\right\}. \tag{1.3}$$

This class is very famous and important in regular function theory and relevant subclasses of it have been obtained by many authors such as ([8]). We note that the case $\mu=0$ reduce to starlike functions of order ξ and the case $\mu=1$ reduce to uniformly starlike functions of order ξ. We also let

$$\mathrm{TC}_p(\xi,\mu)=\mathrm{T}\cap\mathrm{C}_p(\xi,\mu)$$

and

$$\mathrm{TC}^*(\xi)=\mathrm{T}\cap\mathrm{C}^*(\xi)$$

Lemma 1.1 *Let* $0\le\xi<1,$ $\mu\ge 0$ *and* $\beta\in\mathrm{R}.$ *Then* $\Re(w)>\mu|w-t|+\xi$ *is equivalent to* $\Re[w(1+\mu e^{i\beta})-\mu te^{i\beta}]>\xi$ *where* ***w*** *and* ***t*** *are arbitrary complex numbers.*

Lemma 1.2 *Let* $\mu\ge 0$ *and* $t\in\mathrm{C}.$ *Then* $\Re(t)>\mu$ *is equivalent to* $|t-(1+\mu)|<|t+(1-\mu)|.$

COEFFICIENT BOUNDS

In this section we introduce an inequality that provide a necessary and sufficient Coefficient for functions in the class $\mathrm{TC}_p(\xi,\mu).$

Theorem 2.1 *Let* $-1\le\xi\le 1,$ $\mu\ge 0$ *and* $\varphi\in\mathrm{TC}_p(\xi,\mu)$ *be in the form (1.2). Then we have*

$$\sum_{\tau=2}^{\infty}(\tau(1+\mu)-(\mu+\xi))\tau a_\tau\le 1-\xi. \tag{2.1}$$

Proof. Let $\varphi \in \mathrm{TC}_p(\xi,\mu)$ be in the form (1.2). By putting $w = 1 + \frac{z\varphi''(z)}{\varphi'(z)}$ in (1.3) and by lemma 1.1, we obtain $\Re(w(1+\mu e^{i\beta}) - \mu e^{i\beta}) \geq \xi$ or

$$\Re\left(\frac{(1+\mu e^{i\beta})\left(1-\sum_{\tau=2}^{\infty}(\tau(\tau-1)+\tau)a_\tau z^{\tau-1}\right)-(\mu e^{i\beta}+\xi)\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}{\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}\right)\geq 0$$

If $z \in \Theta$ is real and tends to $\mathbf{1}^-$ through reals, then we have

$$\Re\left(1-\xi+\sum_{\tau=2}^{\infty}(\xi-\tau)\tau a_\tau+\mu e^{i\beta}\sum_{\tau=2}^{\infty}(1-\tau)\tau a_\tau\right)\geq 0.$$

Therefore

$$1-\xi-\sum_{\tau=2}^{\infty}(\tau-\xi)\tau a_\tau-\mu e^{i\beta}\sum_{\tau=2}^{\infty}(\tau-1)\tau a_\tau \geq 0.$$

Theorem 2.2 *Let* $\mu \geq 0$ *and* $\varphi \in \mathrm{T}$ *be an analytic function of the form (1.2). Then the following condition is sufficient for* φ *to be in the class* $\mathrm{C}_p(\xi,\mu)$.

$$\sum_{\tau=2}^{\infty}(\mu(\tau-1)+\tau-\xi)\,|a_\tau| \leq 1 \tag{2.2}$$

if $-1 \leq \xi < 0$ and

$$\sum_{\tau=2}^{\infty}(\mu(\tau-1)+\tau-\xi)\,|a_\tau| \leq 1-\xi \tag{2.3}$$

Proof. By lemma 1.1, we note that the condition (1.3) is equivalent to $\Re\left(w\left(1+\mu e^{i\beta}\right)-\left(\xi+\mu e^{i\beta}\right)\right)\geq 0$ where $w = 1+\frac{z\varphi''(z)}{\varphi'(z)}$. So by lemma 1.2, it is sufficient to show that $A \geq B$ where

$$A = \left|1+w\left(1+\mu e^{i\beta}\right)-\left(\xi+\mu e^{i\beta}\right)\right|$$

$$= \frac{\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)+\left(1+\mu e^{i\beta}\right)\left(1-\sum_{\tau=2}^{\infty}\tau^2 a_\tau z^{\tau-1}\right)+\left(\mu e^{i\beta}+\xi\right)\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}{\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}$$

and

$$B = \left|1-w\left(1+\mu e^{i\beta}\right)+\left(\xi+\mu e^{i\beta}\right)\right|$$

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$$= \frac{\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)-\left(1+\mu e^{i\beta}\right)\left(1-\sum_{\tau=2}^{\infty}\tau^2 a_\tau z^{\tau-1}\right)-\left(\mu e^{i\beta}+\xi\right)\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}{\left(1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right)}$$

Let $M = \frac{1}{\left|1-\sum_{\tau=2}^{\infty}\tau a_\tau z^{\tau-1}\right|}$. Therefore

$$A \geq M\left(|2-\xi| - \sum_{\tau=2}^{\infty}\left(\mu(\tau-1)+|1+\tau-\xi|\right)\tau|a_\tau|\right) \tag{2.4}$$

and

$$B \leq M\left(|\xi| + \sum_{\tau=2}^{\infty}\left(\mu(\tau-1)+|\tau-1-\xi|\right)\tau|a_\tau|\right). \tag{2.5}$$

So by the hypothesis, if $-1 \leq \xi < 0$, then by (2.4) and (2.5)

$$A-B \geq 2M\left(1-\sum_{\tau=2}^{\infty}\left(\mu(\tau-1)+\tau-\xi\right)\tau|a_\tau|\right).$$

The last expression is non-negative by (2.2) and so φ belongs to the class $C_p(\xi,\mu)$. Also if $0 \leq \xi \leq 1$, then by (2.4) and (2.5) we obtain

$$A-B \geq 2M\left(1-\xi-\sum_{\tau=2}^{\infty}\left(\mu(\tau-1)+\tau-\xi\right)\tau|a_\tau|\right).$$

The last expression is non-negative by (2.3) and so $\varphi \in C_p(\xi,\mu)$.

The case $\mu = 0$ in two previous theorems leads to

Corollary 2.3 *Let* $\varphi(z) = z - \sum_{\tau=2}^{\infty} a_\tau z^\tau \in T$ *and* $0 \leq \xi \leq 1$. *Then* $\varphi \in C^*(\xi)$ *if and only if*

$\sum_{\tau=2}^{\infty}(\tau-\xi)\tau a_\tau \leq 1-\xi.$

Theorem 2.4 *Let* $-1 \leq \xi \leq 1$, $0 \leq \mu < 1$ *and let* $\varphi_1(z) = z$,

$$\varphi_\tau(z) = z - \frac{1-\xi}{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau} z^z, \quad \tau \geq 2.$$

If $\varphi \in TC_p(\xi,\mu)$ then we have $\varphi(z) = \sum_{\tau=2}^{\infty}\lambda_\tau\varphi_\tau(z)$ where $\lambda_\tau \geq 0$ and $\sum_{\tau=2}^{\infty}\lambda_\tau = 1$

Proof. Let $\varphi \in TC_p(\xi,\mu)$ has the form $z - \sum_{\tau=2}^{\infty} a_\tau z^\tau$. By Theorem 2.1 we obtain

$$\sum_{\tau=2}^{\infty}\frac{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau}{1-\xi} a_\tau \leq 1$$

and

$$a_\tau \leq \frac{1-\xi}{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau}, \quad \tau \geq 2.$$

Therefore we can set $\lambda_\tau = \frac{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau}{1-\xi} a_\tau$ for $\tau = 2,3,\ldots$ and $\lambda_1 = 1-\sum_{\tau=2}^{\infty}\lambda_\tau$. Thus,

$0 \leq \lambda_\tau \leq 1$ for each $\tau \in \mathbb{N}$ and $\sum_{\tau=2}^{\infty}\lambda_\tau = 1$. Also $\boldsymbol{\varphi(z)}$ has the form

$$\varphi(z) = z - \sum_{\tau=2}^{\infty} a_\tau z^\tau = z - \sum_{\tau=2}^{\infty}\frac{\lambda_\tau(1-\xi)}{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau} z^\tau$$

$$= \lambda_1 z + \sum_{\tau=2}^{\infty}\lambda_\tau\left(z - \frac{1-\xi}{\left(\tau(1+\mu)-(\mu+\xi)\right)\tau} z^\tau\right)$$

$$= \sum_{\tau=1}^{\infty}\lambda_\tau \varphi_\tau(z)$$

Theorem 2.5 *Let* $-1 \leq \xi \leq 1$, $0 \leq \mu < 1$. *Also let* $l, m \in \mathbb{C} - \{0\}$ *and* $n > |l| + |m| + 1$. *Then the condition*

$$\frac{\Gamma(n-|l|-|m|-1)\Gamma(n)}{\Gamma(n-|l|)\Gamma(n-|m|)}((1+k)|lm| + (1-\xi)(n-|l|-|m|-1)) \leq 2(1-\xi) \qquad (2.6)$$

if sufficient for the function $z\mathrm{F}(l,m,n;z)$ belongs to $\mathrm{C}_p(\xi,\mu)$.

Proof. Set $z\mathrm{F}(l,m,n;z)$. By Theorem 2.2, we need to show that

$$N := \sum_{\tau=2}^{\infty}[\tau(1+\mu)-(\mu+\xi)]\left|\frac{(l)_{\tau-1}(m)_{\tau-1}}{(n)_{\tau-1}(1)_{\tau-1}}\right| \leq 1-\xi.$$

According to $|(l)_\tau| \leq (|a|)_\tau$, we observe that

$$N \leq \sum_{\tau=2}^{\infty}[\tau(1+\mu)-(\mu+\xi)]\frac{(|l|)_{\tau-1}(|m|)_{\tau-1}}{(n)_{\tau-1}(1)_{\tau-1}}$$

$$= (1+\mu)\sum_{\tau=1}^{\infty}\frac{(\tau+1)(|l|)_\tau(|m|)_\tau}{(n)_\tau(1)_\tau} - (\mu+\xi)\sum_{\tau=1}^{\infty}\frac{(|l|)_\tau(|m|)_\tau}{(n)_\tau(1)_\tau}$$

$$= (1+\mu)\sum_{\tau=1}^{\infty}\frac{(|l|)_\tau(|m|)_\tau}{(n)_\tau(1)_{\tau-1}} + (1-\xi)\sum_{\tau=1}^{\infty}\frac{(|l|)_\tau(|m|)_\tau}{(n)_\tau(1)_\tau}$$

$$= \frac{|lm|}{n}(1+\mu)\sum_{\tau=0}^{\infty}\frac{(1+|l|)_\tau(1+|m|)_\tau}{(1+n)_\tau(1)_\tau} + (1-\xi)\sum_{\tau=1}^{\infty}\frac{(|l|)_\tau(|m|)_\tau}{(n)_\tau(1)_\tau}$$

$$= \frac{|lm|}{n}(1+\mu)\mathrm{F}(1+|l|, 1+|m|, 1+n; 1) + (1-\xi)(\mathrm{F}((|l|),(|m|),n;1)-1)$$

$$= \frac{|lm|}{n}(1+\mu)\frac{\Gamma(n+1)\Gamma(n-|l|-|m|-1)}{\Gamma(n-|l|)\Gamma(n-|m|)} + (1-\xi)\frac{\Gamma(n)\Gamma(n-|l|-|m|)}{\Gamma(n-|l|)\xi(n-|m|)}$$

$$-(1-\xi).$$

Therefore according to (2.6), $\boldsymbol{N}$ is less than $1-\xi$.

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